

Phase transition and thermodynamic geometry of Einstein-Maxwell-dilaton black holes

S. H. Hendi^{1,2*}, A. Sheykhi^{1,2†}, S. Panahiyan^{1‡} and B. Eslam Panah^{1§}

¹ *Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran*

² *Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), Maragha, Iran*

In this paper, we consider a linearly charged dilatonic black holes and study their thermodynamical behavior in the context of phase transition and thermodynamic geometry. We show that, depending on the values of the parameters, these type of black holes may enjoy two types of phase transition. We also find that there are three critical behaviors near the critical points for these black holes; nonphysical unstable to physical stable, large to small, and small to large black holes phase transition. Next, we employ a thermodynamical metric for studying thermodynamical geometry of these black holes. We show that the characteristic behavioral of Ricci scalar of this metric enables one to recognize the type of phase transition and critical behavior of the black holes near phase transition points. Finally, we will extend thermodynamical space by considering dilaton parameter as extensive parameter. We will show that by this consideration, Weinhold, Ruppeiner and Quevedo metrics provide extra divergencies which are not related to any phase transition point whereas our new method is providing an effective machinery.

I. INTRODUCTION

Recent astronomical observations indicate that our Universe is currently undergoing a phase of accelerated expansion [1]. In the context of standard cosmology, based on Einstein gravity, this acceleration cannot be explained unless an unknown energy component usually dubbed “dark energy” is proposed. Another way for explanation of such an acceleration is the modification of the Einstein theory of gravity. In this regards, various modifications of Einstein gravity proposed in the literatures. Among them are Lovelock gravity [2], braneworld scenario [3], scalar-tensor theories [4, 5], $f(R)$ gravity [6], etc.

The studies on the black hole as a thermodynamic system date back to the work of Hawking and Bekenstein [7]. According to the black holes thermodynamics, the geometrical quantities such as horizon area and surface gravity are related to the thermodynamic quantities such as entropy and temperature. The first law of black hole thermodynamics implies that the entropy and the temperature together with the energy (mass) of the black hole satisfy $dE = TdS$ [7]. In recent years, the investigations on the thermodynamical properties of the black holes have got a lot of interests. In particular, thermodynamic properties of black holes in anti de-Sitter (adS) spaces are improved in an extended phase space in which the cosmological constant and its conjugate variable are considered as thermodynamic pressure and volume, respectively [8–10]. In addition, there has been some proposals to consider constants (such as Born-Infeld nonlinearity parameter, Gauss-Bonnet parameter, Newton constant and etc.) as thermodynamical variable which contributes to thermodynamical behavior of the system [11]. It was shown that considering these constants as thermodynamical variables will enrich the phase structure of the black holes and describe new and interesting phenomena such as Van der Waals like liquid/gas behavior. In this paper, motivated by these reasons, we will extend the phase structure of dilatonic black holes by considering dilaton parameter as a thermodynamical extensive parameter.

Recently, there has been several attempts in studying the phase transition in dynamical context. It was shown that the quasinormal modes of a perturbed black hole near critical point, exhibits different behaviors. In Ref. [12], it was argued that due to existence of normal modes only for massless BTZ black holes, there is a phase transition from non-rotating BTZ black holes. On the other hand, the four dimensional topological black holes with scalar hair present the signature of the phase transition in their quasi normal modes [13]. Some evidences regarding second order phase transition of a topological black hole to hairy one are given in Ref. [14]. Furthermore, a dramatic change in the slopes of quasinormal frequencies in small and large black holes near the critical point was observed for four-dimensional Reissner-Nordström-adS black holes [15].

One of the interesting aspects of the black hole thermodynamics is stability of black holes. In order to a black hole to be in thermal stability, its heat capacity must be positive. This approach of studying the stability is in the

* email address: hendi@shirazu.ac.ir

† email address: asheykhi@shirazu.ac.ir

‡ email address: ziexify@gmail.com

§ email address: behzad_eslampanah@yahoo.com

context of the canonical ensemble. Also, studying the heat capacity of a system provides a machinery to study the phase transitions of the black holes. There are two kinds of phase transitions; in the first case, changing in the signature of the heat capacity is denoted as a type of phase transition and therefore, the roots of the heat capacity are phase transition points. So, we call these phase transitions type one. Another type of phase transition is related to divergencies of the heat capacity. This kind of phase transition is called the type two phase transition.

On the other hand, applying thermodynamical geometry to investigate the phase transition of black holes has gained lots of attentions. The pioneering studies were done by Weinhold [16] and Ruppeiner [17]. Weinhold proposed a metric on the space of equilibrium state, introduced as the second derivatives of internal energy with respect to entropy and other extensive quantities. The metric that Ruppeiner introduced was defined as the minus second derivatives of entropy with respect to the internal energy and other extensive quantities. It is worth noting that the Ruppeiner's metric was conformal to Weinhold's metric with the inverse of the temperature as the conformal factor [18]. On the other hand, neither Weinhold nor Ruppeiner metrics were invariant under Legendre transformation. Then, Quevedo [19] proposed an approach to obtain a metric which is Legendre invariant in the space of equilibrium state. His motivation was based on the fact that the standard thermodynamics is invariant under Legendre transformation. The formalism of geometrical thermodynamics (GTs) implies that phase transition occurs at points where the thermodynamics curvature is singular. As a consequence the curvature can be interpreted as a measure of thermodynamic interaction [20]. Recently, it was shown that employing Quevedo metrics for studying geometrothermodynamics may not always lead to effective machinery. In other words, there may be mismatch between phase transition points and divergencies of the Ricci scalar of the Quevedo's metric. To solve the problem, a new metric was proposed which has consistent results with heat capacity [21].

The outline of this paper is as follow. In the next section, we shall review charged dilaton black holes and their thermodynamic quantities. In section III, we will introduce the approaches for studying phase transitions of these black holes in the context of heat capacity and GTs and study the stability of these black holes. Then, we continue our paper and investigate the existence of the phase transitions in the context of two mentioned approaches. We also introduce some concepts such as stability, phase transition and GTs of black holes in this section. In section IV, we discuss the obtained diagrams of the phase transition, heat capacity and GTs of the dilaton black holes. In section V, we consider dilatonic parameter, α as an extensive parameter and investigate the effects of it on different approaches of GTs. The last section is devoted to closing remarks.

II. CHARGED BLACK HOLE SOLUTIONS IN DILATON GRAVITY

The $(n + 1)$ -dimensional action of Einstein-Maxwell-dilaton (EMd) gravity has the following form [22]

$$I = \frac{1}{16\pi} \int d^{n+1}x \sqrt{-g} \left[\mathcal{R} - \frac{4}{n-1} (\nabla\Phi)^2 - V(\Phi) - e^{-4\alpha\Phi/(n-1)} F_{\mu\nu} F^{\mu\nu} \right], \quad (1)$$

where \mathcal{R} is the Ricci scalar curvature, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor and A_μ is the electromagnetic potential, Φ is the dilaton field and $V(\Phi)$ is a potential for Φ . Here, α is a constant determining the strength of coupling of the scalar and electromagnetic field. One can vary action (1) with respect to the gravitational field $g_{\mu\nu}$, the dilaton field Φ and the gauge field A_μ , which leads to the following field equations, respectively,

$$R_{\mu\nu} = \frac{4}{n-1} \left(\partial_\mu \Phi \partial_\nu \Phi + \frac{1}{4} g_{\mu\nu} V(\Phi) \right) + 2e^{-4\alpha\Phi/(n-1)} \left(F_{\mu\eta} F_\nu^\eta - \frac{1}{2(n-1)} g_{\mu\nu} F_{\lambda\eta} F^{\lambda\eta} \right), \quad (2)$$

$$\nabla^2 \Phi = \frac{n-1}{8} \frac{\partial V}{\partial \Phi} - \frac{\alpha}{2} e^{-4\alpha\Phi/(n-1)} F_{\lambda\eta} F^{\lambda\eta}, \quad (3)$$

$$\nabla_\mu \left(e^{-4\alpha\Phi/(n-1)} F^{\mu\nu} \right) = 0. \quad (4)$$

We consider an $(n + 1)$ -dimensional spacetime with the following line element

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 R^2(r) h_{ij} dx^i dx^j, \quad (5)$$

where $f(r)$ and $R(r)$ are functions of r which should be determined, and $h_{ij} dx^i dx^j$ is the line element for an $(n - 1)$ -dimensional subspace with $n(n - 1)k$ constant curvature and volume ω_{n-1} . We should note that the constant k

indicates that the boundary of $t = \text{constant}$ and $r = \text{constant}$ can be a positive (elliptic), zero (flat) or negative (hyperbolic) constant curvature hypersurface. The Maxwell equation (4) can be integrated immediately to give [22]

$$F_{tr} = \frac{qe^{4\alpha\Phi/(n-1)}}{(rR)^{n-1}}. \quad (6)$$

We consider the dilaton potential of the form [22]

$$V(\Phi) = \frac{k(n-1)(n-2)\alpha^2}{b^2(\alpha^2-1)} e^{\frac{4\Phi}{\alpha(n-1)}} + 2\Lambda e^{\frac{4\alpha\Phi}{n-1}}, \quad (7)$$

where Λ is a free parameter which plays the role of the cosmological constant. For later convenience, we redefine it as $\Lambda = -n(n-1)/2l^2$, where l is a constant with dimension of length. It is notable that, this kind of potential was previously investigated by a number of authors both in the context of Friedman-Robertson-Walker (FRW) scalar field cosmologies [23] and EMd black holes [24].

On the other hand, in order to solve the system of equations (2) and (3) for three unknown functions $f(r)$, $R(r)$ and $\Phi(r)$, we make the ansatz [25]

$$R(r) = e^{2\alpha\Phi/(n-1)}. \quad (8)$$

Using Eqs. (6)–(8) and the metric (5), one can easily show that equations (2) and (3) have solutions of the form [22]

$$\begin{aligned} f(r) = & -\frac{k(n-2)(\alpha^2+1)^2}{(\alpha^2-1)(\alpha^2+n-2)} \left(\frac{b}{r}\right)^{-2\gamma} - \frac{m}{r^{(n-1)(1-\gamma)-1}} + \frac{2\Lambda(\alpha^2+1)^2 r^2}{(n-1)(\alpha^2-n)} \left(\frac{b}{r}\right)^{2\gamma} \\ & + \frac{2q^2(\alpha^2+1)^2}{(n-1)(\alpha^2+n-2)r^{2(n-2)}} \left(\frac{b}{r}\right)^{-2(n-2)\gamma}, \end{aligned} \quad (9)$$

$$\Phi(r) = \frac{n-1}{2(\alpha^2+1)} \ln\left(\frac{b}{r}\right), \quad (10)$$

where b is an arbitrary constant and $\gamma = \alpha^2/(\alpha^2+1)$. In the above expression, m and q are integration constants which are related to the total mass and electric charge of the black hole, respectively.

It is notable that, in the absence of a non-trivial dilaton ($\alpha = \gamma = 0$), the solution (9) reduces to

$$f(r) = k - \frac{m}{r^{n-2}} - \frac{2\Lambda}{n(n-1)} r^2 + \frac{2q^2}{(n-1)(n-2)r^{2(n-2)}}, \quad (11)$$

which describes an $(n+1)$ -dimensional asymptotically adS topological black hole with a positive, zero or negative constant curvature hypersurface [26].

A. Thermodynamic quantities of EMd black holes

Since our aim in the next section is to study GTs and heat capacity of EMd black holes, therefore, here we give a brief review regarding the calculation of the conserved charges and thermodynamic quantities in which they have been obtained in [22]. It was shown that by using the Hamiltonian approach, one can find the mass M and charge Q of the EMd black holes as [22]

$$M = \frac{(n-1)b^{(n-1)\gamma}\omega_{n-1}}{16\pi(\alpha^2+1)} m, \quad (12)$$

$$Q = \frac{\omega_{n-1}}{4\pi} q. \quad (13)$$

The Hawking temperature calculated for the topological Emd black hole on the outer horizon r_+ has the following form [22]

$$T = -\frac{k(n-2)(\alpha^2+1)}{2\pi(\alpha^2+n-2)r_+} \left(\frac{b}{r_+}\right)^{-2\gamma} - \frac{(n-\alpha^2)m}{4\pi(\alpha^2+1)} r_+^{(n-1)(1-\gamma)} - \frac{q(\alpha^2+1)}{\pi(\alpha^2+n-2)r_+^{2n-3}} \left(\frac{b}{r_+}\right)^{-2(n-2)\gamma}, \quad (14)$$

where r_+ satisfy $f(r=r_+)=0$. One can obtain the entropy of the topological Emd black hole by employing the area law. One finds [22]

$$S = \frac{b^{(n-1)\gamma}\omega_{n-1}}{4} r_+^{(n-1)(1-\gamma)}. \quad (15)$$

III. THERMAL STABILITY, PHASE TRANSITION AND GTS

In this section, first, we study stability and phase transition of the Emd black holes by considering the heat capacity of the solutions. Next, we consider the GTs approach for studying phase transitions. We investigate the effects of dilaton parameter and compare the results of both approaches.

There are several approaches for studying stability of the black holes. One of these approaches is related to studying the perturbed black holes and see if and how they acquire stable state and will be in equilibrium. This approach is known as dynamical stability of black holes. In this paper, we are not interested in dynamical stability of black holes but thermal one. We focus our studies on thermal stability of charged black hole solutions in the context of Emd theory through canonical ensemble. To do so, we calculate the heat capacity and study its behavior.

Black holes should have a positive heat capacity in order to be thermally stable. In other words, the positivity of the heat capacity ensures the local thermal stability of the black holes. One can use the following relation for the heat capacity

$$C_Q = T \left(\frac{\partial^2 M}{\partial S^2} \right)_Q^{-1} = T \left(\frac{\partial S}{\partial T} \right)_Q = T \left(\frac{\partial S}{\partial r_+} \right)_Q \left(\frac{\partial T}{\partial r_+} \right)_Q^{-1}. \quad (16)$$

On the other hand, one can use heat capacity for studying phase transitions of black holes. In context of black holes, it is argued that roots of the heat capacity are representing a type of phase transition. We call it type one phase transition. The reasoning for such an assumption is due to fact that system in case of this phase transition has a changing in signature of the heat capacity. In other words, unstable system with negative heat capacity goes under a phase transition and its heat capacity changes from negative to positive. In addition, it is believed that the divergencies of the heat capacity are also representing phase transitions of black holes. We call these phase transitions, type two phase transitions. Therefore, the phase transition points of the black holes in context of the heat capacity are calculated with following relations

$$\begin{cases} T = \left(\frac{\partial M}{\partial S} \right)_Q = 0, & \text{Type one} \\ \left(\frac{\partial^2 M}{\partial S^2} \right)_Q = 0, & \text{Type two} \end{cases}. \quad (17)$$

Another approach for studying the phase transition of the black holes is through GTs. There are several metrics that one can employ in order to build geometrical spacetime by thermodynamical quantities. The well known ones are Ruppeiner, Weinhold and Quevedo. It was previously argued that these metrics may not provide us a completely flawless machinery for study GTs of specific types of black holes [21]. Recently, a new metric (HPEM metric) was proposed in order to solve the problems that other metrics may confront [21]. The denominator of the Ricci scalar of HPEM metric only contains numerator and denominator of the heat capacity. In other words, divergence points of the Ricci scalar of HPEM metric coincide with both types of phase transitions of the heat capacity. The metric is in the following form

$$ds_{New}^2 = \frac{SM_S}{\left(\prod_{i=2}^n \frac{\partial^2 M}{\partial \chi_i^2} \right)^3} \left(-M_{SS} dS^2 + \sum_{i=2}^n \left(\frac{\partial^2 M}{\partial \chi_i^2} \right) d\chi_i^2 \right), \quad (18)$$

where $M_S = \partial M / \partial S$, $M_{SS} = \partial^2 M / \partial S^2$ and χ_i ($\chi_i \neq S$) are extensive parameters. In what follows, we will study the stability and phase transition of the charged dilatonic black holes in context of the heat capacity and GTs.

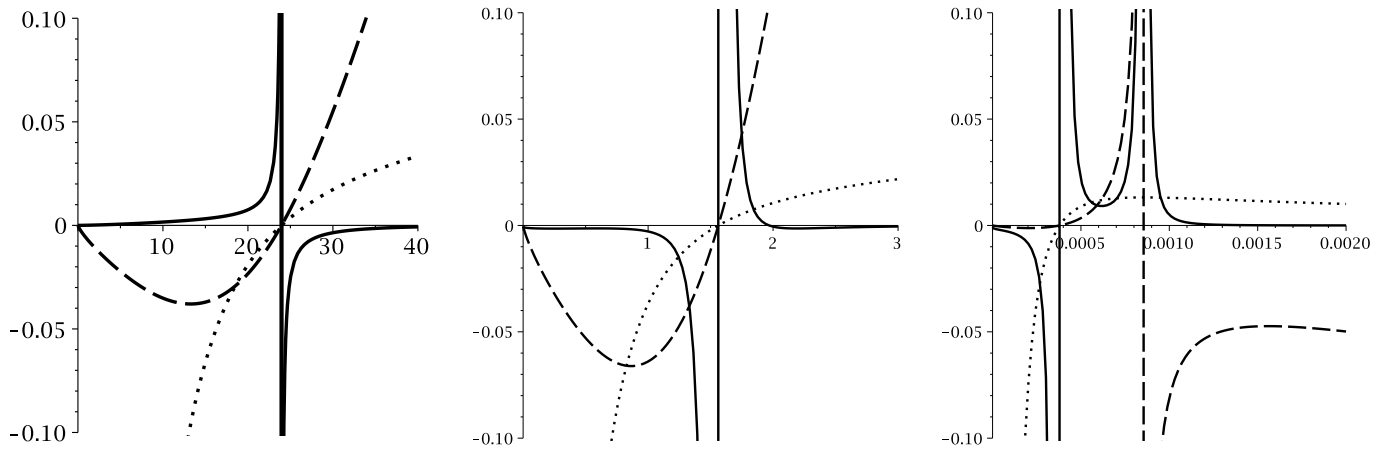


FIG. 1: \mathcal{R} (continuous line), C_Q (dashed line) and T (dotted line) versus r_+ for $k = 1$, $q = 1$, $\alpha = 2$ and $n = 5$ $b = 0.05$ (left), $b = 0.5$ (middle) and $b = 5$ (right)

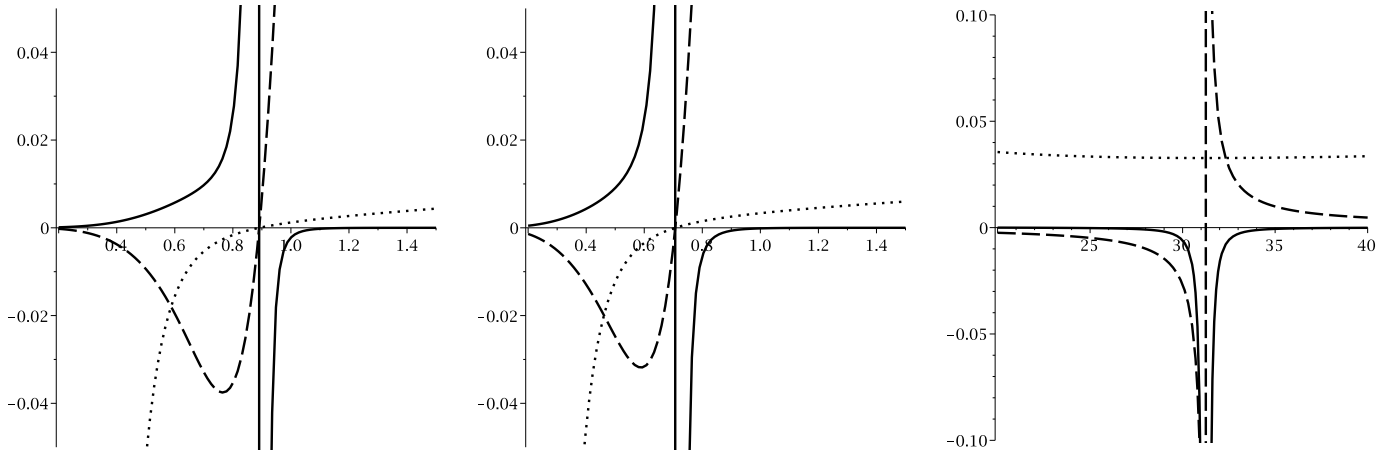


FIG. 2: \mathcal{R} (continuous line), C_Q (dashed line) and T (dotted line) versus r_+ for $k = 1$, $q = 1$, $b = 2$ and $n = 5$ $\alpha = 0.05$ (left), $\alpha = 0.5$ (middle) and $\alpha = 5$ (right).

IV. DIAGRAMS AND DISCUSSIONS

In order to study the effects of different parameters on thermodynamical behavior of the system we have plotted several diagrams in which the effects of variations of b , α and q are taken into account (Figs. 1 to 6).

First we will study the effects of the b . It is evident from plotted graphs that for sufficiently small values of b , there is only type one phase transition (Fig. 1 left and middle). In other words, the denominator of the heat capacity is nonzero and there is only one root for heat capacity. In place of this root, temperature and heat capacity changes sign from negative to positive. Therefore, for the case of $r_+ < r_0$ (in which $C_Q(r_+ = r_0) = 0$), obtained solutions are nonphysical unstable ones. It must be pointed out that, there is only one divergency for Ricci scalar which coincides with root of the heat capacity.

For sufficiently large values of b these black holes will enjoy one phase transition type one and one phase transition type two (Fig. 1 right). Therefore, the system will go under two type of phase transition. If we denote the place of the divergency of the heat capacity with r_∞ , one can see that only for the region of $r_0 < r_+ < r_\infty$, solutions are representing stable black holes. As for $r_+ > r_\infty$, system is unstable state. In this case too, the phase transition points of the heat capacity and divergencies of the heat capacity coincide. It is worthwhile to mention that both phase transition points of the heat capacity are decreasing functions of b .

Next, we are taking the effects of variation of dilation parameter, α into account. Evidently, similar to the case of b , for small values of α , there is only one phase transition of type one, which its place is a decreasing function of α (Fig. 2 left and middle). Interestingly, for sufficiently large values of α , the phase transition type one of the heat

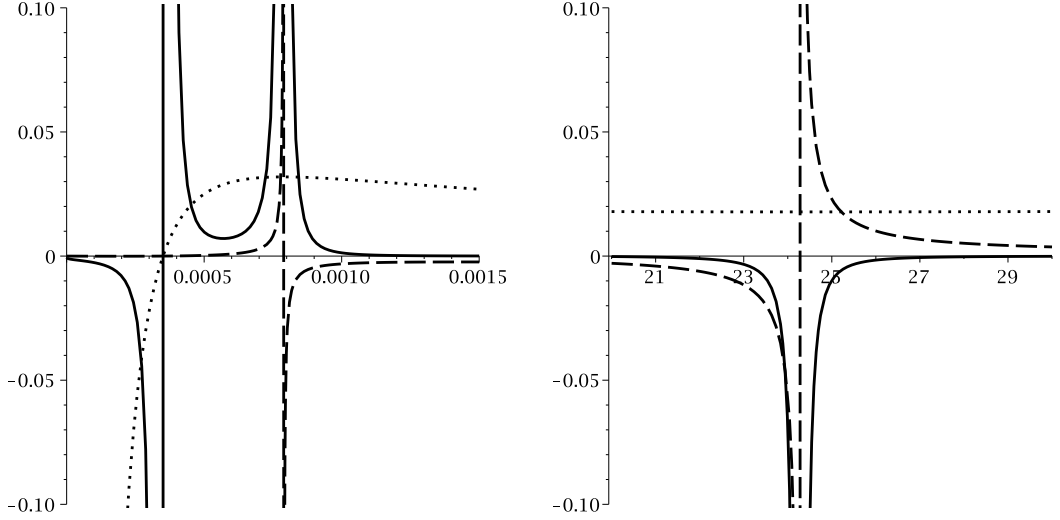


FIG. 3: \mathcal{R} (continuous line), C_Q (dashed line) and T (dotted line) versus r_+ for $k = 1$, $\alpha = 2$, $b = 2$ and $n = 5$ for different scales: $q = 0.05$.

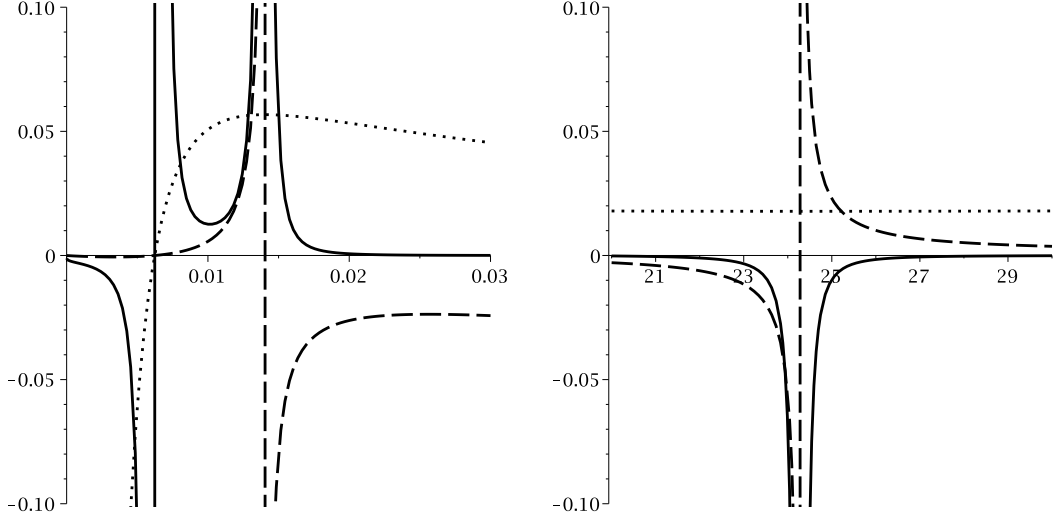


FIG. 4: \mathcal{R} (continuous line), C_Q (dashed line) and T (dotted line) versus r_+ for $k = 1$, $\alpha = 2$, $b = 2$ and $n = 5$ for different scales: $q = 0.5$.

capacity will vanish and there will be only one phase transition type two for these black holes (Fig. 2 right). In this case for $r_+ < r_\infty$, black holes are unstable whereas for $r_+ = r_\infty$, they go under phase transition and acquire stable state. The place of divergency of the heat capacity is an increasing function of α . It is worthwhile to mention that in these cases too, Ricci scalar of the considered thermodynamical metric has divergencies exactly in place of phase transitions of heat capacity.

Our next effects of interest are the ones that are due to variation of charge. As one can see, there are two phase transitions type two and one phase transition type one for considered values for different parameters (Figs. 3 to 5). The root and smaller divergence point of heat capacity are highly increasing functions of charge whereas the larger divergence point of heat capacity is not effected considerably by variation of charge.

Considering the number of the phase transition points, we have three phase transition. In case of $r_+ = r_0$, the temperature and heat capacity changing sign from unstable nonphysical black holes to stable physical ones. Next phase transition takes place in $r_+ = r_S$ (r_S is smaller divergence point of heat capacity). Due to thermodynamical behavior (phase transition from unstable state to stable one), there is a phase transition from black holes with larger horizon radius to ones with smaller horizon radius. In other words, in this place there is a phase transition of larger/smaller black holes. In opposite, in case of phase transition that takes place in $r_+ = r_L$ (r_L is larger divergence point of heat

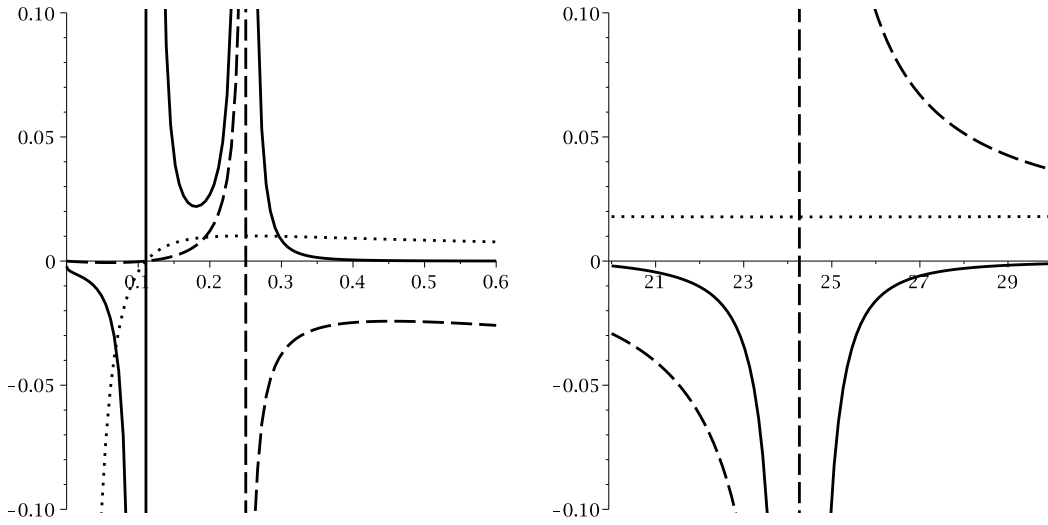


FIG. 5: \mathcal{R} (continuous line), C_Q (dashed line) and T (dotted line) versus r_+ for $k = 1$, $\alpha = 2$, $b = 2$ and $n = 5$ for different scales: $q = 5$.

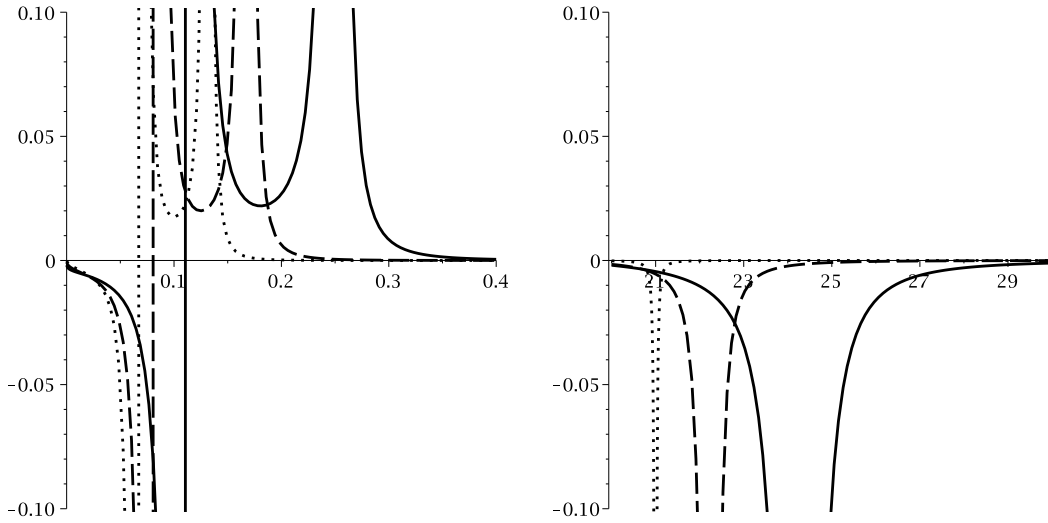


FIG. 6: \mathcal{R} versus r_+ for $k = 1$, $\alpha = 2$, $b = 2$ and $q = 5$ for different scales: $n = 5$ (continuous line), $n = 6$ (dashed line) and $n = 7$ (dotted line) .

capacity) there is a phase transition of smaller/larger black holes. In other words, unstable black holes goes under phase transition and obtain stable state with larger horizon radius.

Finally, as for dimensions, it is evident that root and divergence points of heat capacity are decreasing functions of dimensionality (Fig. 6). It is worthwhile to mention that divergence points of Ricci scalar of considered metric, in all the cases, coincide with phase transition points of heat capacity. The behavior of the Ricci scalar near these divergence points is different for each one of these phase transition points. In case of root of heat capacity, there is a change in signature of the Ricci scalar near corresponding divergency (Figs. 3 to 5 left). As for phase transition of larger/smaller black holes, the signature of the Ricci scalar is fixed and the divergency is toward $+\infty$ (Figs. 3 to 5 left) whereas in case of smaller/larger phase transition the signature is fixed but it is toward $-\infty$ (Figs. 3 to 5 right). Therefore, one is able to recognize type of phase transition and the behavior of it only by studying Ricci scalar of HPEM metric (Fig. 6).

V. EXTENDING THERMODYNAMICAL SPACE

In this section we would like to extend our phase space by considering the dilaton-electromagnetic coupling parameter, α , as an extensive parameter. We shall explore the effects of this phase space extension in different approaches of GTs. Therefore, our thermodynamical space will change from $M(S, Q)$ to $M(S, Q, \alpha)$. This consideration causes the Weinhold, Ruppeiner, Quevedo and HPEM metrics modified into following forms

$$ds^2 = \begin{cases} ds_W^2 = Mg_{ab}^W dX^a dX^b & \text{Weinhold} \\ ds_R^2 = -T^{-1} Mg_{ab}^R dX^a dX^b & \text{Ruppeiner} \\ (SM_S + QM_Q + \alpha M_\alpha) (-M_{SS}dS^2 + M_{QQ}dQ^2 + M_{\alpha\alpha}d\alpha^2) & \text{Quevedo Case I} \\ SM_S (-M_{SS}dS^2 + M_{QQ}dQ^2 + M_{\alpha\alpha}d\alpha^2) & \text{Quevedo Case II} \\ S \frac{M_S}{M_{QQ}^3 M_{\alpha\alpha}^3} (-M_{SS}dS^2 + M_{QQ}dQ^2 + M_{\alpha\alpha}d\alpha^2) & \text{HPEM} \end{cases} \quad (19)$$

It is a matter of calculation to show that mentioned metric with these specific structures have following denominators for their Ricci scalars

$$\text{denom}(\mathcal{R}) = \begin{cases} -2M^3 (M_{Q\alpha}^2 M_{SS} + M_{SQ}^2 M_{\alpha\alpha} + M_{S\alpha}^2 M_{QQ} - M_{SS} M_{QQ} M_{\alpha\alpha} - 2M_{SQ} M_{S\alpha} M_{Q\alpha})^2 & \text{Weinhold} \\ -2M^3 T^3 (M_{Q\alpha}^2 M_{SS} + M_{SQ}^2 M_{\alpha\alpha} + M_{S\alpha}^2 M_{QQ} - M_{SS} M_{QQ} M_{\alpha\alpha} - 2M_{SQ} M_{S\alpha} M_{Q\alpha})^2 & \text{Ruppeiner} \\ 2M_{SS}^2 M_{QQ}^2 M_{\alpha\alpha}^2 (SM_S + QM_Q + \alpha M_\alpha)^3 & \text{Quevedo I} \\ 2S^3 M_{SS}^2 M_{QQ}^2 M_{\alpha\alpha}^3 & \text{Quevedo II} \\ 2S^3 M_{SS}^2 M_S^3 & \text{HPEM} \end{cases} \quad (20)$$

It is evident that, in general, for Weinhold case, due to obtained relation, the coincidence of different types of the phase transition points and divergencies of the Ricci scalar may not take place (Fig. 7 left). As for the Ruppeiner metric, the existence of T , in denominator of the Ricci scalar ensures the coincidence of divergency of the Ricci scalar and phase transition type one. As for the type two, due to other terms of the denominator of the Ricci scalar, in general, the coincidence may not happen or extra divergencies may be observed (Fig. 7 right).

As for Quevedo metric case *II*, due to terms M_{QQ}^2 and $M_{\alpha\alpha}^2$, there may be divergencies which may not be related to any phase transition point of the heat capacity. In other words, these two terms may have roots and their roots will contribute to number of divergencies of the Ricci scalar (Fig. 8 up diagrams). In case of Quevedo case *I*, $SM_S + QM_Q + \alpha M_\alpha$ may have contribution to number of divergencies of the Ricci scalar of this metric. The existence of these divergencies are seen through following plotted graphs (Fig. 8 down diagrams).

Finally, in case of HPEM metric, the denominator of the Ricci scalar of this metric constructed in a way that there is no extra divergence point for its Ricci scalar and all divergencies of the Ricci scalar and phase transition points of the heat capacity will coincide. Therefore, this formalism is providing a machinery with consisting results of studying heat capacity (Fig. 9).

VI. CLOSING REMARKS

In this paper, we have considered the thermodynamical behavior of the topological charged dilaton black holes of EMd gravity. We have investigated the phase transition points related to heat capacity and GTs of the EMd black holes. We have shown that according to the variation of parameter under consideration, the type and number of the phase transitions, stability conditions, thermodynamical behavior of these black holes near critical points will be modified. As for phase transitions, it was shown that these black holes enjoys three thermodynamical behavior near critical points which are nonphysical stable to physical stable, larger/smaller and smaller/larger black holes. We have also pointed out that considering HPEM metric for constructing thermodynamical spacetime leads to effective machinery for studying phase transition points of the heat capacity. In other words, divergencies of the Ricci scalar coincided with phase transition points of heat capacity. It was also shown that Ricci scalar behavior near these critical

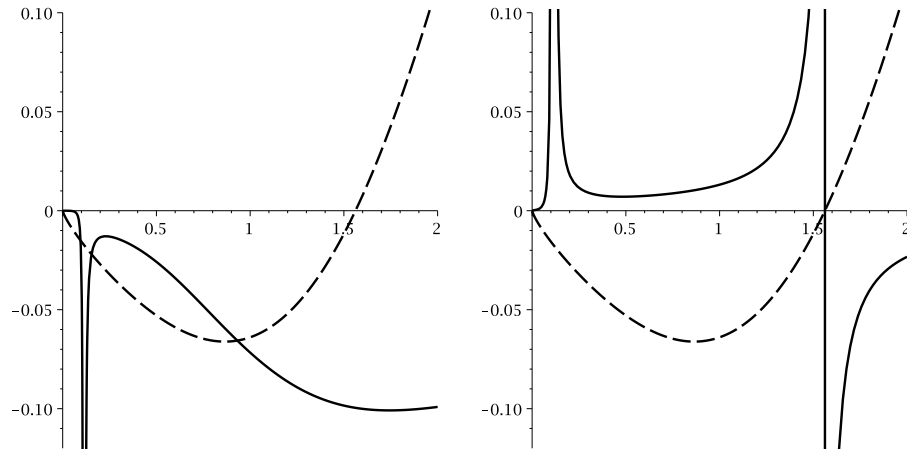


FIG. 7: \mathcal{R} (continues line) and C_Q (dashed line) versus r_+ for $l = 1$, $\Lambda = -1$, $n = 5$ and $q = 1$, $\alpha = 2$, $b = 0.5$ and $k = 1$.
 left diagram: The Weinhold metric.
 right diagram: The Ruppeiner metric.

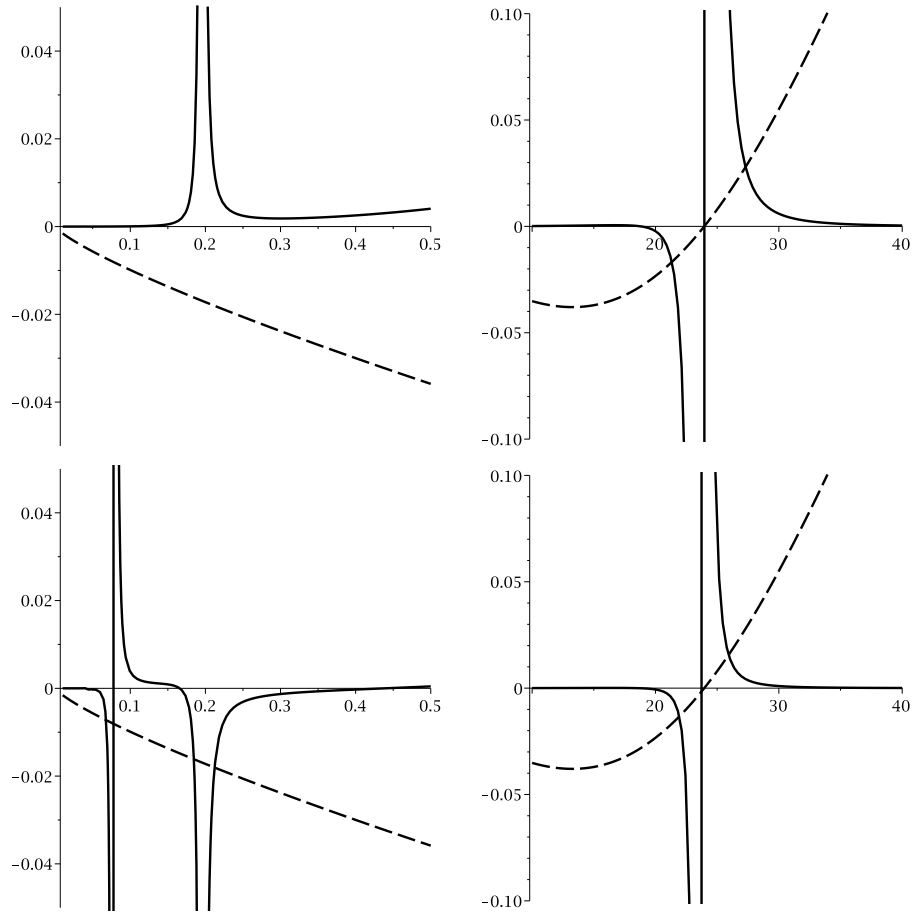


FIG. 8: \mathcal{R} (continues line) and C_Q (dashed line) versus r_+ for $l = 1$, $\Lambda = -1$, $n = 5$ and $q = 1$, $\alpha = 2$, $b = 0.05$ and $k = 1$.
 up diagrams for different scale: The Quevedo metric for case II.
 down diagrams for different scale: The Quevedo metric for case I.

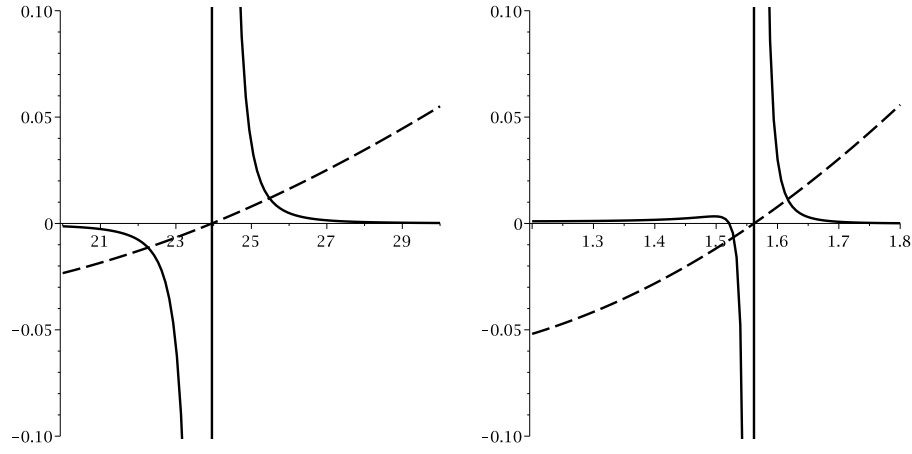


FIG. 9: \mathcal{R} (continues line) and C_Q (dashed line) versus r_+ for $l = 1$, $\Lambda = -1$, $n = 5$ and $q = 1$, $\alpha = 2$ and $k = 1$. left diagram: $b = 0.05$, right diagram: $b = 0.5$.

points depends on thermodynamical behavior near critical point. Therefore, one is able to recognize the type of phase transition and its behavior only by employing HPEM metric.

Next, we considered dilaton parameter as an extensive parameter and extended thermodynamical space. We conducted an study with this consideration for different approaches for GTs. We showed that in order to GTs methods and heat capacity have consisting results, certain conditions must be satisfied otherwise there might be extra divergencies. Only HPEM method was free of any conditions. Later, It was shown that Weinhold, Ruppeiner and Quevedo methods and their Ricci scalars contain divergencies which were not related to any phase transition points of heat capacity. In other words, extra divergencies were seen which were not consistent with results that were obtained in case of heat capacity. Therefore, the mentioned conditions were not hold for these pictures.

It is worthwhile to consider various models of nonlinear electrodynamics instead of Maxwell field in this regard and examine the effects of nonlinearity and also the validity of different thermodynamical metrics. We left this issue for the forthcoming work.

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- [1] S. Perlmutter et al., *Astrophys. J.* **517**, 565 (1999);
S. Perlmutter, M. S. Turner and M. White, *Phys. Rev. Lett.* **83**, 670 (1999);
A. G. Riess et al., *Astrophys. J.* **607**, 665 (2004).
 - [2] D. Lovelock, *J. Math. Phys.* **12**, 498 (1971);
D. Lovelock, *J. Math. Phys.* **13**, 874 (1972);
N. Deruelle and L. Farina-Busto, *Phys. Rev. D* **41**, 3696 (1990);
S. H. Hendi and M. H. Dehghani, *Phys. Lett. B* **666**, 116 (2008).
 - [3] P. Brax and C. van de Bruck, *Class. Quantum Gravit.* **20**, R201 (2003);
L. A. Gergely, *Phys. Rev. D* **74**, 024002 (2006);
M. Demetrian, *Gen. Relativ. Gravit.* **38**, 953 (2006);
L. Amarilla and H. Vucetich, *Int. J. Mod. Phys. A* **25**, 3835 (2010).
 - [4] P. Jordan, *Schwerkraft und Weltall* (Friedrich Vieweg und Sohn, Brunschweig, 1955);
C. Brans and R. H. Dicke, *Phys. Rev.* **124**, 925 (1961);
Yasunori Fujii and Kei-ichi Maeda. *The Scalar-Tensor Theory of Gravitation*. Cambridge (GB): University Press (2003);
T. P. Sotiriou, *Class. Quantum Gravit.* **23**, 5117 (2006).
 - [5] S. B. Giddings and A. Strominger, *Phys. Rev. D* **47**, 2454 (1993);
R. Gregory and C. Santos, *Phys. Rev. D* **56**, 1194 (1997);
P. Klepac and J. Horský, *Gen. Relativ. Gravit.* **34**, 1979 (2002);

- R. G. Cai, S. P. Kim and B. Wang, Phys. Rev. D **76**, 024011 (2007);
Y. Ling, C. Niu, J. P. Wu and Z. Y. Xian, JHEP **11**, 006 (2013);
M. Ghodrati, Phys. Rev. D **90**, 044055 (2014).
- [6] J. C. C. de Souza and V. Faraoni, Class. Quantum Gravit. **24**, 3637 (2007);
K. Bamba and S. D. Odintsov, JCAP **04**, 024 (2008);
G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, L. Sebastiani and S. Zerbini, Phys. Rev. D **77**, 046009 (2008);
C. Corda, Europhys. Lett. **86**, 20004 (2009);
T. P. Sotiriou and V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010);
S. Nojiri and S. D. Odintsov, Phys. Rept. **505**, 59 (2011);
S. H. Hendi, B. Eslam Panah and S. M. Mousavi, Gen. Relativ. Gravit. **44**, 835 (2012);
S. H. Hendi, R. B. Mann, N. Riazi and B. Eslam Panah, Phys. Rev. D **86**, 104034 (2012);
A. Sheykhi, Phys. Rev. D **86**, 024013 (2012);
S. H. Hendi, B. Eslam Panah and R. Saffari, Int. J. Mod. Phys. D **23**, 1450088 (2014).
- [7] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973);
S. W. Hawking, Nature **248**, 30 (1974).
- [8] M. Henneaux and C. Teitelboim, Phys. Lett. B **143**, 415 (1984);
M. Henneaux and C. Teitelboim, Phys. Lett. B **222**, 195 (1989);
C. Teitelboim, Phys. Lett. B **158**, 293 (1985).
- [9] Y. Sekiwa, Phys. Rev. D **73**, 084009 (2006);
A. Larranaga, [arXiv:0711.0012];
H. Quevedo and A. Sanchez, JHEP **09**, 034 (2008).
- [10] J. D. Brown and C. Teitelboim, Phys. Lett. B **195**, 177 (1987);
J. D. Brown and C. Teitelboim, Nucl. Phys. B **297**, 787 (1988);
M. Cvetič, S. Nojiri and S. D. Odintsov, Nucl. Phys. B **628**, 295 (2002);
- [11] G. W. Gibbons, R. Kallosh, and B. Kol, Phys. Rev. Lett. **77**, 4992 (1996);
D. A. Rasheed, [arXiv:hep-th/9702087];
N. Breton, Gen. Relativ. Gravit. **37**, 643 (2005);
D. Kastor, S. Ray, and J. Traschen, Class. Quantum Gravit. **26**, 195011 (2009);
W. Y. Huan, Chin. Phys. B **19**, 090404 (2010);
B. P. Dolan, Class. Quantum Gravit. **28**, 125020 (2011);
B. P. Dolan, Class. Quantum Gravit. **28**, 235017 (2011);
D. Kubiznak and R. B. Mann, JHEP **07**, 033 (2012);
S. H. Hendi and M. H. Vahidinia, Phys. Rev. D **88**, 084045 (2013);
R. G. Cai, L. M. Cao, L. Li and R. Q. Yang, JHEP **09**, 005 (2013);
D. C. Zou, Y. Liu and B. Wang, Phys. Rev. D **90**, 044063 (2014);
B. P. Dolan, D. Kastor, D. Kubiznak, R. B. Mann, and J. Traschen, Phys. Rev. D **87**, 104017 (2014);
W. Xu, H. Xu and L. Zhao, Eur. Phys. J. C, **74**, 2970 (2014);
D. C. Zou, S. J. Zhang and B. Wang, Phys. Rev. D **89**, 044002 (2014);
S. H. Hendi, S. Panahiyan and R. Mamasani, Gen. Relativ. Gravit. **47**, 91 (2015).
- [12] X. Rao, B. Wang and G. Yang, Phys. Lett. B **649**, 472 (2007).
- [13] J. Shen, B. Wang, C. Y. Lin, R. G. Cai and R. K. Su, JHEP **07**, 037 (2007).
- [14] G. Koutsoumbas, S. Musiri, E. Papantonopoulos and G. Siopsis, JHEP **10**, 006 (2006).
- [15] Y. Liu, D. C. Zou and B. Wang, JHEP **09**, 179 (2014).
- [16] F. Weinhold, J. Chem. Phys. **63**, 2479 (1975);
F. Weinhold, J. Chem. Phys. **63**, 2484 (1975).
- [17] G. Ruppeiner, Phys. Rev. A **20**, 1608 (1979);
G. Ruppeiner, Rev. Mod. Phys. **67**, 605 (1995).
- [18] P. Salamon, J. D. Nulton and E. Ihrig, J. Chem. Phys. **80**, 436 (1984).
- [19] H. Quevedo, Gen. Relativ. Gravit. **40**, 971 (2008);
H. Quevedo, A. Sanchez, S. Taj and A. Vazquez, Gen. Relativ. Gravit. **43**, 1153 (2011);
A. Bravetti, D. Momeni, R. Myrzakulov and H. Quevedo, Gen. Relativ. Gravit. **45**, 1603 (2013).
- [20] R. G. Cai and J. H. Cho, Phys. Rev. D **60**, 067502 (1999);
J. E. Aman, I. Bengtsson and N. Pidokrajt, Gen. Relativ. Gravit. **35**, 1733 (2003);
J. E. Aman and N. Pidokrajt, Phys. Rev. D **73**, 024017 (2006).
- [21] S. H. Hendi, S. Panahiyan, B. Eslam Panah and M. Momennia, [arXiv:1506.08092];
S. H. Hendi, S. Panahiyan, B. Eslam Panah and M. Momennia, [arXiv:1506.07262];
S. H. Hendi, S. Panahiyan and B. Eslam Panah, [arXiv:1410.0352];
S. H. Hendi, S. Panahiyan and B. Eslam Panah, Adv. High Energy Phys. **2015**, 743086 (2015);
S. H. Hendi, S. Panahiyan, B. Eslam Panah and Z. Armanfard, *Phase transition of charged Black Holes in Brans-Dicke theory through geometrothermodynamics, submitted for publication.*
- [22] A. Sheykhi, Phys. Rev. D **76**, 124025 (2007).
- [23] M. Ozer and M. O. Taha, Phys. Rev. D **45**, 997 (1992);
R. Easther, Class. Quantum Gravit. **10**, 2203 (1993).
- [24] K. C. K. Chan, J. H. Horne and R. B. Mann, Nucl. Phys. B **447**, 441 (1995);

- S. S. Yazadjiev, *Class. Quantum Gravit.* **22**, 3875 (2005);
A. Sheykhi and N. Riazi, *Phys. Rev. D* **75**, 024021 (2007).
[25] M. H. Dehghani and N. Farhangkhah, *Phys. Rev. D* **71**, 044008 (2005).
[26] D. R. Brill, J. Louko and P. Peldan, *Phys. Rev. D* **56**, 3600 (1997);
R. G. Cai and K. S. Soh, *Phys. Rev. D* **59**, 044013 (1999).